Exercise 18

The figure shows a pendulum with length L and the angle θ from the vertical to the pendulum. It can be shown that θ, as a function of time, satisfies the nonlinear differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

where g is the acceleration due to gravity. For small values of θ we can use the linear approximation $\sin\theta \approx \theta$ and then the differential equation becomes linear.

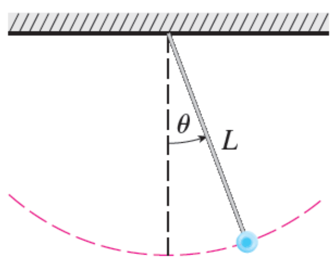
(a) Find the equation of motion of a pendulum with length 1 m if θ is initially 0.2 rad and the initial angular velocity is $d\theta/dt = 1$ rad/s.

(b) What is the maximum angle from the vertical?

(c) What is the period of the pendulum (that is, the time to complete one back-and-forth swing)?

(d) When will the pendulum first be vertical?

(e) What is the angular velocity when the pendulum is vertical?



Solution

For small angles, Newton's second law can be linearized as follows.

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0, \quad \theta(0) = \theta_0, \quad \frac{d\theta}{dt}(0) = \omega_0$$

This is a linear homogeneous ODE, so its solutions are of the form $\theta = e^{rt}$.

$$\theta = e^{rt} \quad \rightarrow \quad \frac{d\theta}{dt} = re^{rt} \quad \rightarrow \quad \frac{d^2\theta}{dt^2} = r^2e^{rt}$$

Plug these formulas into the ODE.

$$r^2e^{rt} + \frac{g}{L}(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + \frac{g}{L} = 0$$

Solve for r.

$$r = \left\{-i\sqrt{\frac{g}{L}}, i\sqrt{\frac{g}{L}}\right\}$$

Two solutions to the ODE are $e^{-i\sqrt{g/L}t}$ and $e^{i\sqrt{g/L}t}$. By the principle of superposition, then,

$$\begin{aligned}
\theta(t) &= C_1 e^{-i\sqrt{g/L}t} + C_2 e^{i\sqrt{g/L}t} \\
&= C_1\left(\cos\sqrt{\frac{g}{L}}t - i\sin\sqrt{\frac{g}{L}}t\right) + C_2\left(\cos\sqrt{\frac{g}{L}}t + i\sin\sqrt{\frac{g}{L}}t\right) \\
&= (C_1 + C_2)\cos\sqrt{\frac{g}{L}}t + (-iC_1 + iC_2)\sin\sqrt{\frac{g}{L}}t \\
&= C_3\cos\sqrt{\frac{g}{L}}t + C_4\sin\sqrt{\frac{g}{L}}t,
\end{aligned}$$

where C_3 and C_4 are arbitrary constants. Differentiate it with respect to t.

$$\frac{d\theta}{dt} = -C_3\sqrt{\frac{g}{L}}\sin\sqrt{\frac{g}{L}}t + C_4\sqrt{\frac{g}{L}}\cos\sqrt{\frac{g}{L}}t$$

Apply the initial conditions to determine C_3 and C_4.

$$\theta(0) = C_3 = \theta_0$$

$$\frac{d\theta}{dt}(0) = C_4\sqrt{\frac{g}{L}} = \omega_0$$

Solving this system yields

$$C_3 = \theta_0 \quad \text{and} \quad C_4 = \omega_0\sqrt{\frac{L}{g}}.$$

The general solution is then

$$\theta(t) = \theta_0\cos\sqrt{\frac{g}{L}}t + \omega_0\sqrt{\frac{L}{g}}\sin\sqrt{\frac{g}{L}}t.$$

Therefore, plugging in $g = 9.81$ m/s^2 and $L = 1$ m and $\theta_0 = 0.2$ rad and $\omega_0 = 1$ rad/s,

$$\boxed{\theta(t) \approx 0.2\cos 3.132t + 0.319\sin 3.132t.}$$

The period is

$$\boxed{T = \frac{2\pi}{\sqrt{\frac{g}{L}}} = 2\pi\sqrt{\frac{L}{g}} \approx 2.01 \text{ s.}}$$

In order to find the maximum angle, write the two terms as one in the formula for $\theta(t)$ by setting $A\cos\delta = \theta_0$ and $A\sin\delta = \omega_0\sqrt{L/g}$.

$$\begin{aligned}
\theta(t) &= A\cos\delta\cos\sqrt{\frac{g}{L}}t + A\sin\delta\sin\sqrt{\frac{g}{L}}t \\
&= A\left(\cos\delta\cos\sqrt{\frac{g}{L}}t + \sin\delta\sin\sqrt{\frac{g}{L}}t\right) \\
&= A\cos\left(\sqrt{\frac{g}{L}}t - \delta\right)
\end{aligned}$$

The amplitude A is the maximum angle from the vertical; solve for it by squaring both sides of each defining equation and adding the respective sides.

$$A^2\cos^2\delta + A^2\sin^2\delta = (\theta_0)^2 + \left(\omega_0\sqrt{\frac{L}{g}}\right)^2$$

$$A^2(\cos^2\delta + \sin^2\delta) = \theta_0^2 + \frac{\omega_0^2 L}{g}$$

$$A^2 = \theta_0^2 + \frac{\omega_0^2 L}{g}$$

Plug in $g = 9.81$ m/s^2 and $L = 1$ m and $\theta_0 = 0.2$ rad and $\omega_0 = 1$ rad/s to get the maximum angle from the vertical.

$$\boxed{A = \sqrt{\theta_0^2 + \frac{\omega_0^2 L}{g}} \approx 0.377 \text{ rad}}$$

Divide the two defining equations to get δ, the phase angle.

$$\frac{A\sin\delta}{A\cos\delta} = \frac{\omega_0\sqrt{L/g}}{\theta_0}$$

$$\tan\delta = \frac{\omega_0}{\theta_0}\sqrt{\frac{L}{g}}$$

$$\delta = \tan^{-1}\left(\frac{\omega_0}{\theta_0}\sqrt{\frac{L}{g}}\right) \approx 1.011 \text{ rad}$$

The pendulum is first vertical when $\theta = 0$ at the smallest positive value of t, that is, when the argument of cosine is $\pi/2$.

$$\sqrt{\frac{g}{L}}t - \delta = \frac{\pi}{2}$$

Solve for t.

$$\boxed{t = \frac{\delta + \frac{\pi}{2}}{\sqrt{\frac{g}{L}}} \approx 0.824 \text{ s}}$$

To find the angular velocity when the pendulum is vertical, take the derivative of $\theta(t)$

$$\theta'(t) = -\theta_0\sqrt{\frac{g}{L}}\sin\sqrt{\frac{g}{L}}t + \omega_0\cos\sqrt{\frac{g}{L}}t$$

and evaluate it at $t \approx 0.824$ s, the time the pendulum is first vertical.

$$\boxed{\theta'(0.824) \approx -1.18\ \frac{\text{rad}}{\text{s}}}$$